# Rational base numeration system for *p*-adic numbers

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# Outline

$$\sum_{k=-\ell}^{n,\infty} \frac{a_i}{q} \left(\frac{p}{q}\right)^i,$$

$a_i \in \mathcal{A}_p = \{0, 1, \ldots, p-1\}, \quad p > q \geq 1$ co-prime

1. Brief introduction to $p$-adic numbers
2. Representation of positive integers [Akiyama, Frougny, Sakarovitch, 2008]
3. Representation of negative integers
4. Representation of rational numbers
5. Finite representations
6. Representations of $p$-adic numbers

# *p*-adic absolute value

## Definition

*Let p be a prime number.*
*The p-adic valuation* $v_p : \mathbb{Z} \setminus \{0\} \to \mathbb{R}$ *is given by*

$$n = p^{v_p(n)} n' \quad \text{with} \quad p \nmid n'.$$

*The extension to the set of rational numbers, for* $x = \frac{a}{b} \in \mathbb{Q}$

$$v_p(x) = v_p(a) - v_p(b).$$

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$$v_p(x) = v_p(a) - v_p(b).$$

*The p-adic absolute value on $\mathbb{Q}$:*

$$|x|_p = \begin{cases} 0 & \text{if } x = 0, \\ p^{-v_p(x)} & \text{otherwise.} \end{cases}$$

# $p$-adic absolute value – examples

## Example

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$$p = r_1^{j_1} r_2^{j_2} \cdots r_k^{j_k}, q \text{ co-prime to } p \quad \Rightarrow$$

$$\left| \left( \frac{p}{q} \right)^n \right|_r = \begin{cases} r_i^{-j_i n} & \text{if } r = r_i, i \in \{1, 2, \ldots, k\}, \\ \geq 1 & \text{otherwise.} \end{cases}$$

# $\mathbb{Q}_p$ – the set of $p$-adic numbers I

The set of $p$-adic numbers $\mathbb{Q}_p$ is defined as a completion of $\mathbb{Q}$ with respect to $|\ |_p$.

## Lemma

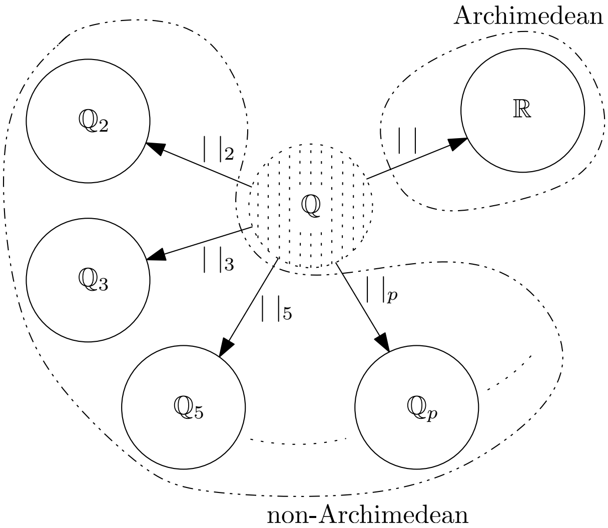
*The $p$-adic absolute value $|\ |_p$ is non-Archimedean, i.e., for all $x, y \in \mathbb{Q}_r$ the following holds:*

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

## Theorem (Ostrowski)

*Every non-trivial absolute value on $\mathbb{Q}$ is equivalent to the classical absolute value $|\ |$ or to one of the absolute values $|\ |_p$, where $p$ is prime.*

# $\mathbb{Q}_p$ – the set of *p*-adic numbers II



# Standard representation of $p$-adic numbers I

## Theorem

*Every $x \in \mathbb{Q}_p$ can be uniquely written as*

$$\begin{aligned} x &= b_{-k_0} p^{-k_0} + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k + \cdots \\ &= \sum_{k \geq -k_0} a_k p^k \end{aligned}$$

*with $a_k \in \mathcal{A}_p = \{0, 1, \ldots, p-1\}$ and $k_0 = -v_p(x)$.*

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## Algorithm

*Let $s \in \mathbb{Z}, s \neq 0$. Put $s_0 = s$ and for all $i \in \mathbb{N}$ define $s_{i+1}$ and $a_i \in \mathcal{A}_p$ by*

$$s_i = ps_{i+1} + a_i.$$

*Return $\mathbf{a} = \cdots a_2 a_1 a_0$.*

# Standard representation of $p$-adic numbers I

## Theorem

*Let $x \in \mathbb{Q}_p$. Then the standard representation of $x$ is*

1. *uniquely given,*
2. *finite if, and only if, $x \in \mathbb{N}$,*
3. *eventually periodic if, and only if, $x \in \mathbb{Q}$.*

# MD algorithm

## Algorithm (MD algorithm)

*Let $s$ be a positive integer. Put $s_0 = s$ and for all $i \in \mathbb{N}$:*

$$qs_i = ps_{i+1} + a_i.$$

*Return $\frac{p}{q}$-expansion of $s$: $< s >_{\frac{p}{q}} = \cdots a_2 a_1 a_0$.*

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$$s = s_0 = s_1 \frac{p}{q} + \frac{a_0}{q} = s_2 \left(\frac{p}{q}\right)^2 + \frac{a_1}{q}\frac{p}{q} + \frac{a_0}{q} = \cdots$$

$$\cdots = s_k \left(\frac{p}{q}\right)^k + \sum_{i=1}^{k-1} \frac{a_i}{q} \left(\frac{p}{q}\right)^i = \cdots = \sum_{i=1}^{n} \frac{a_i}{q} \left(\frac{p}{q}\right)^i$$

# $\frac{p}{q}$-expansion – properties

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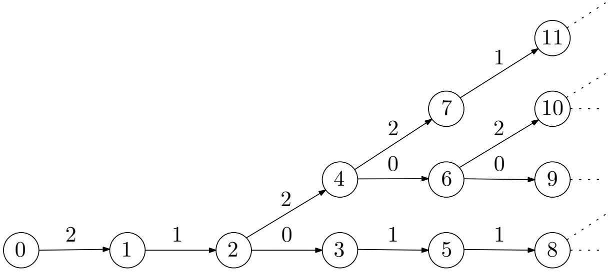
$$L_{\frac{p}{q}} = \{w \in \mathcal{A}_p^* \mid w \text{ is } \frac{p}{q}\text{-expansion of some } s \in \mathbb{N}\}$$

- $L_{\frac{p}{q}}$ is prefix-closed,
- any $u \in \mathcal{A}_p^+$ is a suffix of some $w \in L_{\frac{p}{q}}$,
- $L_{\frac{p}{q}}$ is not context-free (if $q \neq 1$),
- $\pi : \mathcal{A}_p^+ \mapsto \mathbb{Q}$ the evaluation map. If $v, w \in L_{\frac{p}{q}}$, then

$$v \preceq w \quad \Leftrightarrow \quad \pi(v) \leq \pi(w).$$

# $T_{\frac{p}{q}}$ – tree of nonnegative integers

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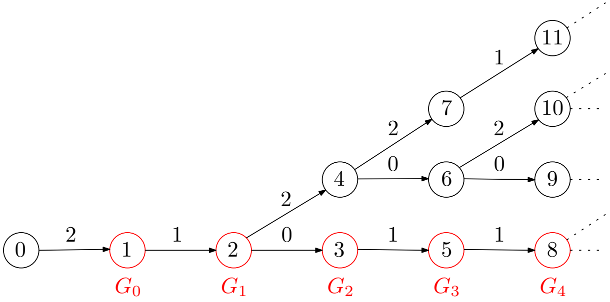


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$$G_0 = 1, \quad G_{i+1} = \left\lceil \frac{p}{q} G_i \right\rceil$$

# MD algorithm – the negative case

Let $s$ be a negative integer. The $\frac{p}{q}$-expansion of $s$ is $< s >_{\frac{p}{q}} = \cdots a_2 a_1 a_0$ from the MD algorithm:

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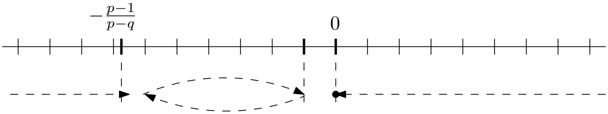
$$s_0 = s, \quad qs_i = ps_{i+1} + a_i.$$

Properties of $(s_i)_{i \geq 0}$:

(i) $(s_i)_{i \geq 1}$ is negative,

(ii) if $s_i < -\frac{p-1}{p-q}$, then $s_i < s_{i+1}$,

(iii) if $-\frac{p-1}{p-q} \leq s_i < 0$, then $-\frac{p-1}{p-q} \leq s_{i+1} < 0$.



# In which fields does it work?

$$s = s_k \left(\frac{p}{q}\right)^k + \sum_{i=1}^{k-1} \frac{a_i}{q} \left(\frac{p}{q}\right)^i$$

We want:

$$\left| s - \sum_{i=1}^{k-1} \frac{a_i}{q} \left(\frac{p}{q}\right)^i \right|_r = |s_k|_r \left| \left(\frac{p}{q}\right)^k \right|_r \to 0 \quad \text{as } k \to \infty$$

Hence, if $p = r_1^{j_1} r_2^{j_2} \cdots r_k^{j_k}$, the $\frac{p}{q}$-expansion of $s$ "works" only in $\mathbb{Q}_{r_i}$, $i = 1, \ldots, k$. The speed of convergence is then $\approx r^{-j_i k}$.

# $\frac{p}{q}$-expansions of negative integers

## Proposition

*Let $k$ be a positive integer, and denote $B = \left\lfloor \frac{p-1}{p-q} \right\rfloor$. Then:*

(i) *if $k \leq B$, then $< -k >_{\frac{1}{q}\frac{p}{q}} = {}^{\omega}b$ with $b = k(p-q)$,*

(ii) *otherwise, $< -k >_{\frac{1}{q}\frac{p}{q}} = {}^{\omega}bw$ with $w \in \mathcal{A}_p^+$ and $b = B(p-q)$.*

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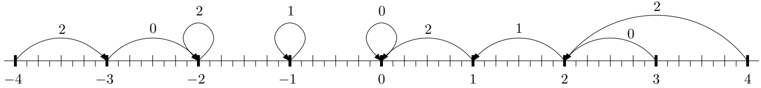
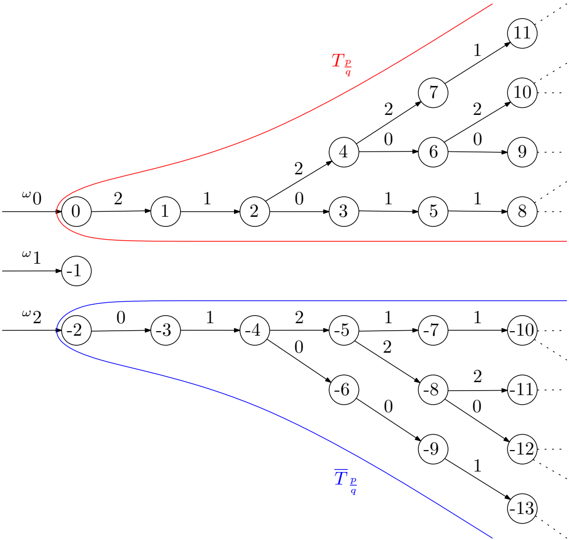


# $\overline{T}_{\frac{p}{q}}$ – tree of negative integers



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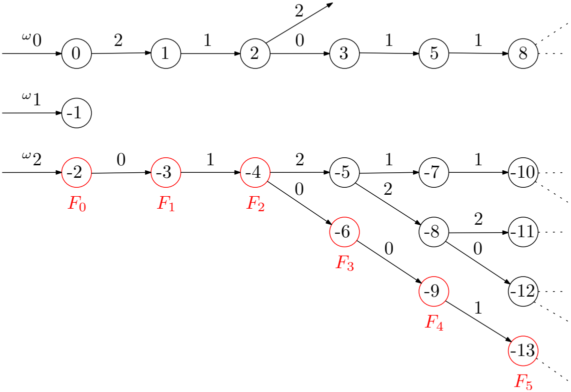
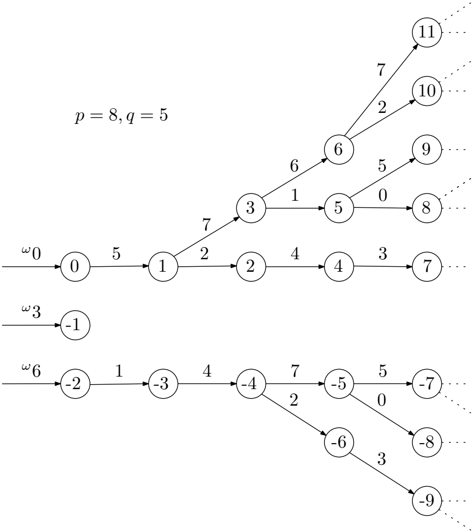



$$F_0 = -B, \quad F_{i+1} = \left\lceil \frac{p}{q} F_i \right\rceil$$

# Trees $\overline{T}_{\frac{p}{q}}$ and $T_{\frac{p}{q}}$



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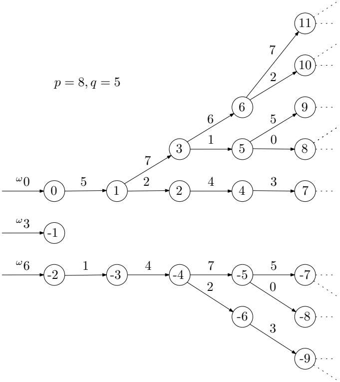



The trees $\overline{T}_{\frac{p}{q}}$ and $T_{\frac{p}{q}}$ are isomorphic if and only if

$$\frac{p-1}{p-q} \in \mathbb{Z}, \quad \text{i.e., } B(p-q) = p-1$$

# $\frac{p}{q}$-expansion – properties

- for $q = 1$ and $p$ prime we get the standard $p$-adic representation,

$$\overline{L}_{\frac{p}{q}} = \{w \in \mathcal{A}_p^* \mid {}^{\omega}bw \text{ is } \frac{p}{q}\text{-expansion of } s \leq -B, w_0 \neq b\}$$

- $\overline{L}_{\frac{p}{q}}$ is prefix-closed,
- any $u \in \mathcal{A}_p^+$ is a suffix of some $w \in \overline{L}_{\frac{p}{q}}$,
- $\overline{L}_{\frac{p}{q}}$ is not context-free (if $q \neq 1$).

# MD algorithm for rationals

## Algorithm (MD algorithm)

*Let* $x = \frac{s}{t}$, $s, t \in \mathbb{Z}$ *co-prime,* $s \neq 0$, *and* $t > 0$ *co-prime to* $p$.
*Put* $s_0 = s$ *and for all* $i \in \mathbb{N}$ *define* $s_{i+1}$ *and* $a_i \in \mathcal{A}_p$ *by*

$$q\frac{s_i}{t} = p\frac{s_{i+1}}{t} + a_i.$$

*Return the* $\frac{p}{q}$*-expansion of* $x$*:* $< x >_{\frac{p}{q}} = \cdots a_2 a_1 a_0$.

# Examples

| $x$ | $<x>_{\frac{1}{q}\frac{p}{q}}$ | $(s_i)_{i\geq 0})$ | abs. values |
|---|---|---|---|
| $p=3, q=2$ | | | |
| 5 | 2101 | 5, 3, 2, 1, 0, 0, ... | all |
| -5 | $^{\omega}$2012 | -5,-3,-2,-2,-2, ... | $\lvert\ \rvert_3$ |
| 11/4 | 201 | 11,6,4,0,0, ... | all |
| 11/8 | $^{\omega}$1222 | 11,2,-4,-8,-8,-8, ... | $\lvert\ \rvert_3$ |
| 11/5 | $^{\omega}$(02)2112 | 11,4,1,-1,-4,-6,-4,-6, ... | $\lvert\ \rvert_3$ |
| $p=30, q=11$ | | | |
| 5 | 11 25 | 5, 1, 0, 0, ... | all |
| -5 | $^{\omega}$19 8 5 | -5,-2,-1,-1,... | $\lvert\ \rvert_2, \lvert\ \rvert_3, \lvert\ \rvert_5$ |
| 11/7 | $^{\omega}$(12 21 5) 23 13 | 11, 1, -5, -3, -6, -5, ... | $\lvert\ \rvert_2, \lvert\ \rvert_3, \lvert\ \rvert_5$ |

# MD algorithm – properties

For the sequence $(s_i)_{i\geq 1}$ from the MD algorithm we have:

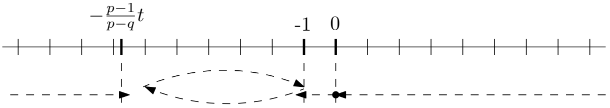
(i) if $s > 0$ and $t = 1$, $(s_i)_{i\geq 1}$ is eventually zero,

(ii) if $s > 0$ and $t > 1$, $(s_i)_{i\geq 1}$ is either eventually zero or eventually negative,

(iii) if $s < 0$, $(s_i)_{i\geq 1}$ is negative,

(iv) if $s_i < -\frac{p-1}{p-q}t$, then $s_i < s_{i+1}$,

(v) if $-\frac{p-1}{p-q}t \leq s_i < 0$, then $-\frac{p-1}{p-q}t \leq s_{i+1} < 0$.



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## Proposition

*Let $x \in \mathbb{Q}$. Then $< x >_{\frac{p}{q}}$ is eventually periodic with period $< \frac{p-1}{p-q}t$.*

# Finite $\frac{p}{q}$-expansion

Let $x$ have finite $\frac{p}{q}$-expansion, then

$$x = \sum_{i=0}^{n} \frac{a_i}{q} \left(\frac{p}{q}\right)^i = \frac{s}{q^{n+1}}, \quad s > 0.$$

Define for all positive integers $m$ the set

$$\mathrm{INF}(m) := \left\{ s \,\middle|\, s > 0, \left\langle \frac{s}{q^m} \right\rangle_{\frac{1}{q}\frac{p}{q}} \text{ is infinite} \right\}.$$

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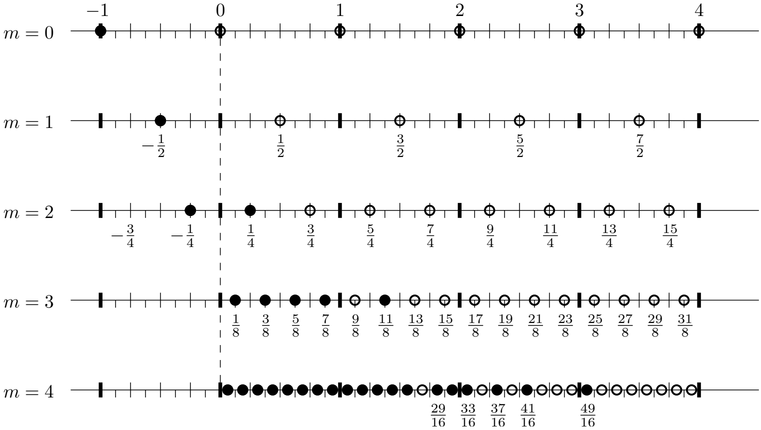
## Example ($p = 3, q = 2$)

$INF(1) = \emptyset, \quad INF(2) = \{1\}, \quad INF(3) = \{1, 2, 3, 5, 7, 11\},$
$INF(4) = \{1, 2, 3, 4, 5, \ldots, 22, 23, 25, 29, 31, 33, 37, 41, 49\},$
$\# INF(5) = 277, \quad \# INF(6) = 1101$

# Finite $\frac{p}{q}$-expansion – II



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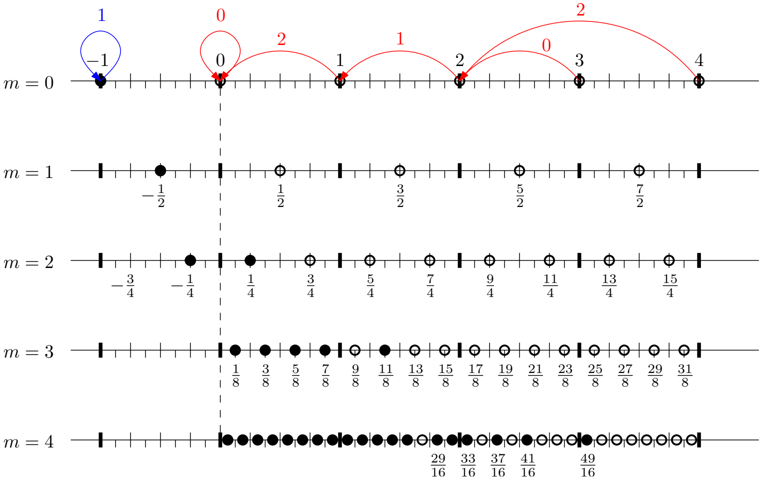

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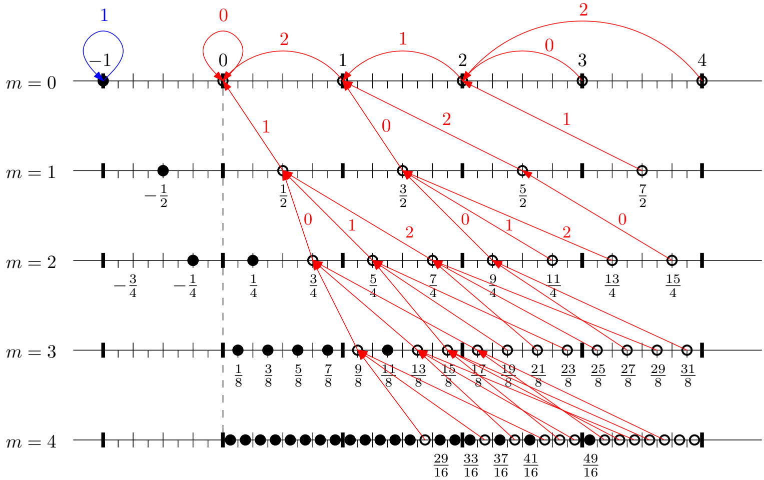

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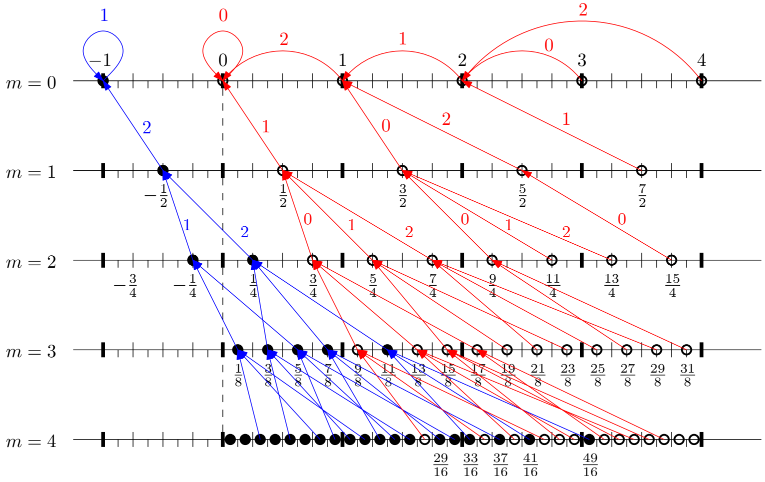

# Alternative algorithm

## Algorithm

*Let* $x = \frac{s}{q^m}$, $s, m$ *positive integers. Put* $h_0 = s$ *and* $h_{i+1}$ *and* $b_i \in \mathcal{A}_p$ *define as follows: for* $i = 0, 1, \ldots, m-1$ *by*

$$\frac{h_i}{q^{m-(i+1)}} = p\frac{h_{i+1}}{q^{m-(i+1)}} + b_i,$$

*for* $i = m, m+1, \ldots$ *by*

$$qh_i = ph_{i+1} + b_i.$$

*Return* $\mathbf{b} = \cdots b_2 b_1 b_0$.

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$$\frac{h_i}{q^{m-(i+1)}} = p\frac{h_{i+1}}{q^{m-(i+1)}} + b_i,$$

*for $i = m, m+1, \ldots$ by*

$$qh_i = ph_{i+1} + b_i.$$

*Return $\mathbf{b} = \cdots b_2 b_1 b_0$.*

It holds that $\mathbf{b} =< x >_{\frac{p}{q}}$ and

$$s_i = \begin{cases} h_i q^i & i = 0, 1, \ldots, m-1 \\ h_i q^m & i = m, m+1, \ldots . \end{cases}$$

# Finite $\frac{p}{q}$-expansions III

## Proposition

*$INF(1) = \emptyset$ and $INF(m) = A(m) \cup B(m)$, $m = 2, 3, \ldots$, where*

$$\begin{aligned} A(m) &= \left\{ -kp + aq^{m-1} \mid k \geq 1, a \in \mathcal{A}_p \right\} \cap \mathbb{N} \\ B(m) &= \left\{ pk + aq^{m-1} \mid k \in F(m-1), a \in \mathcal{A}_p \right\}. \end{aligned}$$

*Moreover, for $m \geq 2$ we get*

$$\frac{\max INF(m)}{q^m} = \frac{p-1}{p-q}\left(\left(\frac{p}{q}\right)^m - 1\right) - \left(\frac{p}{q}\right)^m,$$

*$\frac{p}{q}$-expansion of this number is ${}^{\omega}(p-q)(p-1)^m$.*

# Finite $\frac{p}{q}$-expansions – example

## Example ($p = 12, q = 5$)

$$
\begin{aligned}
INF(1) &= \emptyset \\
INF(2) &= \{1, 2, 3, 4, 6, 7, 8, 9, 11, 13, 14, \ldots, 26, 28, 31, 33, 38, 43\} \\
\#INF(3) &= 1480, \\
\#INF(4) &= 26100, \\
\#INF(5) &= 403549.
\end{aligned}
$$

# $\frac{p}{q}$-representation I

## Definition

*A left infinite word $\cdots a_{-\ell_0+1}a_{-\ell_0}, \ell_0 \in \mathbb{N}$, over $\mathcal{A}_p$ is a $\frac{p}{q}$-representation of $x \in \mathbb{Q}_r$ if $a_{-\ell_0} > 0$ or $\ell_0 = 0$ and*

$$x = \sum_{k=-\ell_0}^{\infty} \frac{a_k}{q} \left(\frac{p}{q}\right)^k$$

*with respect to $| \, |_r$.*

If the $\frac{p}{q}$-representation is not finite (i.e., does not begin in infinitely many zeros), then $r$ is a prime factor of $p$.

# $\frac{p}{q}$-representation II

Let $p = r_1^{j_1} r_2^{j_2} \cdots r_k^{j_k}$.

## Theorem

*Let $x \in \mathbb{Q}_{r_i}$ for some $i \in \{1, \ldots, k\}$.*

(i) *If $k = 1$, there exists a unique $\frac{p}{q}$-representation of $x$ in $\mathbb{Q}_{r_i}$.*

(ii) *If $k > 1$, there exist uncountably many $\frac{p}{q}$-representations of $x$ in $\mathbb{Q}_{r_i}$.*

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(i) *If $k = 1$, there exists a unique $\frac{p}{q}$-representation of $x$ in $\mathbb{Q}_{r_i}$.*

(ii) *If $k > 1$, there exist uncountably many $\frac{p}{q}$-representations of $x$ in $\mathbb{Q}_{r_i}$.*

*Let $x \in \mathbb{Q}$ and let $i_1, \ldots, i_\ell \in \{1, \ldots, k\}, \ell < k$, be distinct.*

(i) *There exist uncountably many $\frac{p}{q}$-representations which work in all fields $\mathbb{Q}_{r_{i_1}}, \ldots, \mathbb{Q}_{r_{i_\ell}}$.*

(ii) *There exists a unique $\frac{p}{q}$-representation which works in all fields $\mathbb{Q}_{r_1}, \ldots, \mathbb{Q}_{r_k}$; namely, the $\frac{p}{q}$-expansion $< x >_{\frac{p}{q}}$.*

(ii) *The $\frac{p}{q}$-expansion $< x >_{\frac{p}{q}}$ is the only $\frac{p}{q}$-representation which is eventually periodic.*