An Algorithm Enumerating All Infinite Repetitions in a D0L System

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D0L system

Definition

A D0L-system is a triplet \( G = (\mathcal{A}, \varphi, w) \) where \( \mathcal{A} \) is an alphabet, \( \varphi \) a morphism on \( \mathcal{A} \), and \( w \in \mathcal{A}^+ \) is the axiom.

The sequence of \( G \):

\[
L(G) = \{ w_0 = w, w_1 = \varphi(w_0), w_2 = \varphi(w_1), \ldots \}.
\]

All factors of \( w_1, w_2, \ldots \) form the language of \( G \), denoted as \( S(L(G)) \).
D0L system

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\]

All factors of \( w_1, w_2, \ldots \) form the **language** of \( G \), denoted as \( S(L(G)) \).

\[G = (\{0, 1, 2, 3, 4\}, \varphi, 013) \text{ with } \varphi = (0310, 212, 121, 4, 3):\]

\[
w_0 = 013
\]
\[
w_1 = 0310 212 4
\]
\[
w_2 = 031041210310 121212121 3
\]
\[
w_3 = 03104121031032121212121212 \ldots 0310 2121212121 \ldots 212 4
\]
\[\vdots\]
Repetitive D0L system

Definition

A D0L system $G$ is repetitive if for all $k \in \mathbb{N}$ there exists a word $v$ such that $v^k$ is in the language of $G$.

It is strongly repetitive if there is a word $v$ such that $v^k$ is in the language of $G$ for all $k$.
Repetitive D0L system

**Definition**

A D0L system $G$ is **repetitive** if for all $k \in \mathbb{N}$ there exists a word $v$ such that $v^k$ is in the language of $G$. 
It is **strongly repetitive** if there is a word $v$ such that $v^k$ is in the language of $G$ for all $k$.

**Example**

The D0L system $(\{0, 1, 2, 3, 4\}, \varphi, 0)$ with $\varphi = (0310, 212, 121, 4, 3)$ is strongly repetitive with $v = 21$. 

Theorem (Ehrenfeucht, Rozenberg (1983))

Every repetitive D0L system is strongly repetitive.
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Every repetitive D0L system is strongly repetitive.
Our motivation

We have a tool (KK 2012) for generating all bispecial factors in a given D0L system, but it works only for non-repetitive D0L systems.

We needed a fast and easy to program algorithm deciding repetitiveness.
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We have a tool (KK 2012) for generating all bispecial factors in a given D0L system, but it works only for non-repetitive D0L systems.

We needed a fast and easy to program algorithm deciding repetitiveness.

We also believe this tool can be extended to repetitive D0L systems but:

We (probably) need to know all the infinite repetitions.
Known results

Theorem (Ehrenfeucht, Rozenberg (1983))

*It is decidable whether a D0L system is repetitive.*
Known results

Theorem (Ehrenfeucht, Rozenberg (1983))

It is decidable whether a D0L system is repetitive.

- If the D0L system is not finite nor pushy, their procedure produces unknown number of special D0L systems.
- The original D0L system is repetitive iff one of these special D0L systems is repetitive.
- A special D0L \((A, \varphi, w)\) system is repetitive iff the morphism is \((B, \pi)\)-cyclic for some \(B \subset A\) and \(\pi\) a cyclic permutation of \(B\):
  - for all \(b \in B\), \(\varphi = x\pi(x)\pi^2(x) \cdots \pi^k(x)\) with \(k = |\varphi(b)| - 1\),
  - for all \(b \in B\): if \(c = \pi(b)\) and \(d\) is the last letter of \(\varphi(b)\), then \(\pi(d)\) is the first letter of \(\varphi(c)\).
Known results

- Mignosi, Séébold (1993): they addressed a different problem, decidability of repetitiveness is just a consequence.

- Kobayashi, Otto (2000): polynomial time algorithm, that still can be simplified.
Related problem: periodicity

Problem (ultimate periodicity)

Given a D0L system \((A, \varphi, w)\) such that \(\varphi(w) = wy\) for some \(y \in A^*\). Is \(\varphi^\omega(w)\) ultimately periodic?

- Other proofs: Honkala (2008), Halava, Harju, Kärki (WORDS 2011).

Among other things a very simple algorithm deciding whether a fixed point of a morphism is purely periodic is presented.

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Related problem: periodicity

Problem (ultimate periodicity)

*Given a D0L system $(A, \varphi, w)$ such that $\varphi(w) = wy$ for some $y \in A^*$. Is $\varphi^\omega(w)$ ultimately periodic?*

- Other proofs: Honkala (2008), Halava, Harju, Kärki (WORDS 2011).

Problem (eventual ultimate periodicity)

*Given a D0L system $(A, \varphi, w)$. Is there $i \geq 0$ and $p > 0$ such that $\varphi^p(w_i) = w_i y$ for some $y \in A^*$ and $(\varphi^p)^\omega(w_i)$ ultimately periodic?*

- First proofs: Head, Lando (1986).
- Refined proof and algorithm: Lando (1989):
  > Among other things a very simple algorithm deciding whether a fixed point of a morphism is purely periodic is presented.
Infinite periodic factor

Since the repetitions

$$(123)^\omega, \ (123123)^\omega, \ (312)^\omega, \ (231)^\omega, \ldots$$

are in fact the same, we define:

**Definition**

Given a D0L system $G$, we say that $v^\omega$ is an **infinite periodic factor** of $G$ if $v$ is a non-empty word and $v^k \in S(L(G))$ for all integers $k$.

Let $v$ be non-empty and primitive (not a power of a shorter word). We say that infinite periodic factors $v^\omega$ and $u^\omega$ are **equivalent** if $u$ is a power of a conjugate of $v$. We denote the equivalence class containing $v^\omega$ by $[v]^\omega$. 
Example

The morphism

\[ \varphi : \ a \to aca, \ b \to badc, \ c \to acab, \ d \to adc \]

is not injective, as \( \varphi(ab) = aca \ badc = acab \ adc = \varphi(cd) \). This means that \{aca, badc, acab, adc\} is not a code.
Simplification of a morphism

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By the defect theorem, there must be three (or less) words \( X, Y, Z \) from \( \{a, b, c, d\}^+ \) such that

\[ \{aca, badc, acab, adc\}^* = \{X, Y, Z\}^*. \]
**Example**

The morphism

\[ \varphi : \begin{align*}
a & \rightarrow aca, \\
b & \rightarrow badc, \\
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\end{align*} \]

is not injective, as \( \varphi(ab) = aca badc = acab adc = \varphi(cd) \). This means that \( \{aca, badc, acab, adc\} \) is not a code.

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\[ g : \begin{align*}
X & \rightarrow b, \\
Y & \rightarrow aca, \\
Z & \rightarrow adc
\end{align*} \]

\[ h : \begin{align*}
a & \rightarrow Y, \\
b & \rightarrow XZ, \\
c & \rightarrow YX, \\
d & \rightarrow Z
\end{align*} \]
Simplification of a morphism

Example

The morphism

\[ \varphi : \quad a \rightarrow \text{aca}, \quad b \rightarrow \text{badc}, \quad c \rightarrow \text{acab}, \quad d \rightarrow \text{adc} \]

is not injective, as \( \varphi(ab) = \text{aca badc} = \text{acab adc} = \varphi(cd) \). This means that \( \{\text{aca, badc, acab, adc}\} \) is not a code.

By the defect theorem, there must be three (or less) words \( X, Y, Z \) from \( \{a, b, c, d\}^+ \) such that

\[ \{\text{aca, badc, acab, adc}\}^* = \{X, Y, Z\}^*. \]

\[ g : \quad X \rightarrow b, \quad Y \rightarrow \text{aca}, \quad Z \rightarrow \text{adc} \]
\[ h : \quad a \rightarrow Y, \quad b \rightarrow XZ, \quad c \rightarrow YX, \quad d \rightarrow Z \]

We have: \( \varphi = g \circ h \) and

injective simplification \( h \circ g : \quad X \rightarrow XZ, \quad Y \rightarrow YYXY, \quad Z \rightarrow YZYX. \)
Simplification of a morphism

Definition

Let $A$ and $B$ be two finite alphabets and let $\varphi: A^* \mapsto A^*$ and $\psi: B^* \mapsto B^*$ be morphisms. We say $\psi$ is a simplification of $\varphi$, if there exist morphisms $h: A^* \mapsto B^*$ and $g: B^* \mapsto A^*$ satisfying $g \circ h = \varphi$ and $h \circ g = \psi$ and $\#B < \#A$. If a morphism has no simplification, it is called elementary.
Simplification of a morphism

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- Elementary morphism is injective.
Simplification of a morphism

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite alphabets and let $\varphi : \mathcal{A}^* \mapsto \mathcal{A}^*$ and $\psi : \mathcal{B}^* \mapsto \mathcal{B}^*$ be morphisms. We say $\psi$ is a simplification of $\varphi$, if there exist morphisms $h : \mathcal{A}^* \mapsto \mathcal{B}^*$ and $g : \mathcal{B}^* \mapsto \mathcal{A}^*$ satisfying $g \circ h = \varphi$ and $h \circ g = \psi$ and $\# \mathcal{B} < \# \mathcal{A}$. If a morphism has no simplification, it is called elementary.

- Elementary morphism is injective.
- D0L system $(\mathcal{A}, \varphi, w)$ is repetitive iff $(\mathcal{B}, \psi, h(w))$ is repetitive.
Simplification of a morphism

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- Elementary morphism is injective.
- D0L system $(A, \varphi, w)$ is repetitive iff $(B, \psi, h(w))$ is repetitive.
- Lando (1989) + our result: There is one-to-one correspondence between infinite periodic factors of these two D0L systems.
Simplification of a morphism

**Definition**

Let $A$ and $B$ be two finite alphabets and let $\varphi : A^* \rightarrow A^*$ and $\psi : B^* \rightarrow B^*$ be morphisms. We say $\psi$ is a simplification of $\varphi$, if there exist morphisms $h : A^* \rightarrow B^*$ and $g : B^* \rightarrow A^*$ satisfying $g \circ h = \varphi$ and $h \circ g = \psi$ and $\#B \leq \#A$. If a morphism has no simplification, it is called elementary.

- Elementary morphism is injective.

- D0L system $(A, \varphi, w)$ is repetitive iff $(B, \psi, h(w))$ is repetitive.

- Lando (1989) + our result: There is one-to-one correspondence between infinite periodic factors of these two D0L systems.

- The construction of a simplification of a non-injective morphism can be done in polynomial time.
Graph of infinite factors

Definition

Let $G = (\mathcal{A}, \varphi, \omega)$ be a D0L-system. The graph of infinite periodic factors of $G$, denoted $P_G$, is a directed graph with loops allowed and defined as follows:

1. the set of vertices of $P_G$ is the set

$$V(P_G) = \{[v]^{\omega} \mid v^\omega \text{ is an infinite periodic factor of } S(L(G))\};$$

2. there is a directed edge from $[v]^{\omega}$ to $[z]^{\omega}$ if $\varphi(v^\omega) \in [z]^{\omega}$.

Obviously, the outdegree of any vertex of $P_G$ is equal to one.
**Lemma**

*If* $G = (A, \varphi, w)$ *is an injective D0L system, then any vertex* $[v]^{\omega} \in P_G$ *has indegree at least 1.*
Graph of infinite factors

**Lemma**

If $G = (\mathcal{A}, \varphi, w)$ is an injective D0L system, then any vertex $[v]^\omega \in P_G$ has indegree at least 1.

**Corollary**

If $G = (\mathcal{A}, \varphi, w)$ is an injective D0L system, then its graph of infinite periodic factors $P_G$ is 1-regular. In other words, $P_G$ consists of disjoint cycles.
Pushy D0L system

**Definition**

Given a morphism \( \varphi \) on \( A \). A letter \( a \in A \) is **bounded** if the language of \((A, \varphi, a)\) is finite; \( A_0 \) is the set of all bounded letters.

**Definition**

A D0L system \( G \) is **pushy**, if its language contains infinite number of factors over \( A_0 \).
Pushy D0L system

Definition

Given a morphism \( \varphi \) on \( \mathcal{A} \). A letter \( a \in \mathcal{A} \) is **bounded** if the language of \( (\mathcal{A}, \varphi, a) \) is finite; \( \mathcal{A}_0 \) is the set of all bounded letters.

Definition

A D0L system \( G \) is **pushy**, if its language contains infinite number of factors over \( \mathcal{A}_0 \).

Example

The D0L system \( (\{0, 1, 2, 3, 4\}, \varphi, 0) \) with \( \varphi = (0310, 212, 121, 4, 3) \). The bounded letters are \( \mathcal{A}_0 = \{3, 4\} \). But it is not pushy.
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A D0L system \( G \) is pushy, if its language contains infinite number of factors over \( A_0 \).

Example

The D0L system \( (\{0, 1, 2, 3, 4\}, \varphi, 0) \) with \( \varphi = (0310, 212, 121, 4, 3) \). The bounded letters are \( A_0 = \{3, 4\} \). But it is not pushy

Example

Consider again the D0L system \( (\{0, 1, 2, 3, 4\}, \varphi, 0) \) with \( \varphi = (03103, 212, 121, 4, 3) \). The bounded letters are \( A_0 = \{3, 4\} \). The system is pushy as \( (34)^k \) is a factor for all \( k \in \mathbb{N} \).
Pushy D0L system: what is known

- It is decidable whether a D0L system is pushy (Ehrenfeucht, Rozenberg (1983)).
  - Pushy iff edge condition: there exist $a \in A$, $k \in \mathbb{N}^+$, $v \in A^*$ and $u \in A_0^+$ such that $\varphi^k(a) = vau$ or $\varphi^k(a) = uav$. 

An algorithm based on a simple graphs.

Graphs on unbounded letters: there is a directed edge from $a$ to $b$ with label $u$ if $\varphi(a) = vbu$ (resp. $\varphi(a) = ubv$) with $v \in A^*$ and $u \in A^*$. 

Pushy iff there is a cycle with a nonempty label.

Theorem (Cassaigne, Nicolas (2010))

If $G$ is a non-erasing pushy D0L system, then there exist $K \in \mathbb{N}$ and a finite set $U$ of words from $A^*_0$ such that every factor from $S(L(G)) \cap A^*_0$ is of one of the following three forms:

1. $w_1$,
2. $w_1u_1w_2$,
3. $w_1u_1w_2u_2w_3$,

where $u_1, u_2 \in U$, $|w_j| < K$ for all $j \in \{1, 2, 3\}$, and $k_1, k_2 \in \mathbb{N}^+$. 

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  - Pushy iff edge condition: there exist \( a \in \mathcal{A} \), \( k \in \mathbb{N}^+ \), \( v \in \mathcal{A}^* \) and \( u \in \mathcal{A}_0^+ \) such that \( \varphi^k(a) = vau \) or \( \varphi^k(a) = uav \).

- An algorithm based on a simple graphs.
  - Graphs on unbounded letters: there is a directed edge from \( a \) to \( b \) with label \( u \) if \( \varphi(a) = vbu \) (resp. \( \varphi(a) = ubv \)) with \( v \in \mathcal{A}^* \) and \( u \in \mathcal{A}_0^* \).
  - Pushy iff there is a cycle with a non empty label.
Pushy D0L system: what is known

- It is decidable whether a D0L system is pushy (Ehrenfeucht, Rozenberg (1983)).
  - Pushy iff **edge condition**: there exist \( a \in A, k \in \mathbb{N}^+, v \in A^* \) and \( u \in A_0^+ \) such that \( \varphi^k(a) = vau \) or \( \varphi^k(a) = uav \).

- An algorithm based on a simple graphs.
  - Graphs on unbounded letters: there is a directed edge from \( a \) to \( b \) with label \( u \) if \( \varphi(a) = vbu \) (resp. \( \varphi(a) = ubv \)) with \( v \in A^* \) and \( u \in A_0^* \).
  - Pushy iff there is a cycle with a non empty label.

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**Theorem (Cassaigne, Nicolas (2010))**

*If \( G \) is a non-erasing pushy D0L system, then there exist \( K \in \mathbb{N} \) and a finite set \( \mathcal{U} \) of words from \( A_0^+ \) such that every factor from \( S(L(G)) \cap A_0^+ \) is of one of the following three forms:*

1. \( w_1 \),
2. \( w_1 u_1^{k_1} w_2 \),
3. \( w_1 u_1^{k_1} w_2 u_2^{k_2} w_3 \),

*where \( u_1, u_2 \in \mathcal{U}, |w_j| < K \) for all \( j \in \{1, 2, 3\} \), and \( k_1, k_2 \in \mathbb{N}^+ \).*
Infinite periodic factors containing an unbounded letter

**Theorem**

If \([v]^{\omega}\) is an infinite periodic factor of a D0L system \(G = (A, \varphi, w)\) such that \(v \notin A_0^+\), then there exist

- \(u\) such that \(u^{\omega}\) is equivalent to \(v^{\omega}\),
- \(a \in A\) and \(\ell \leq \#A\) such that \(u^{\omega}\) is the fixed point of \(\varphi^\ell\) starting with \(a\).

In other words: all infinite periodic factors containing an unbounded letter are purely periodic.
Theorem

If $[v]^\omega$ is an infinite periodic factor of a D0L system $G = (\mathcal{A}, \varphi, w)$ such that $v \not\in \mathcal{A}_0^+$, then there exist

- $u$ such that $u^\omega$ is equivalent to $v^\omega$,
- $a \in \mathcal{A}$ and $\ell \leq \#\mathcal{A}$ such that $u^\omega$ is the fixed point of $\varphi^\ell$ starting with $a$.

In other words: all infinite periodic factors containing an unbounded letter are purely periodic periodic points of $\varphi$. 
**Problem**: for a morphism \( \varphi \) over \( \mathcal{A} \), letter \( a \in \mathcal{A} \) and integer \( \ell \) such that \( \varphi^\ell(a) = av \) with \( v \in \mathcal{A}^+ \) decide whether \( (\varphi^\ell)\infty(a) \) is purely periodic:

1. If \( v \in \mathcal{A}_0^+ \), return the result: \( (\varphi^\ell)\omega(a) \) is not purely periodic (but eventually periodic).
2. Apply \( \varphi^\ell \) to \( a \) until \( (\varphi^\ell)^k(a) \) contains two occurrences of one unbounded letter \( (k < \#\mathcal{A}) \).
3. If this letter is not \( a \), then \( (\varphi^\ell)\omega(a) \) is not periodic, if it is, denote \( u \) the longest prefix containing \( a \) only as the first letter.
4. Now, \( (\varphi^\ell)\omega(a) \) is periodic if and only if \( \varphi^\ell(u) = u^m \) for some integer \( m \geq 2 \).
A note about the algorithm by Ehrenfeucht and Rozenberg

Corollary

Let $G = (A, \varphi, w)$ with $\varphi$ injective and $A_0 = \emptyset$. It holds that $G$ is repetitive iff $\varphi$ is $(B, \pi)$-cyclic for some $B \subseteq \text{alph}(S(L(G)))$ and $\pi$ a cyclic permutation of $B$.

Proof.

- There must be a primitive $u$ and $\ell \geq 1$ such that $\varphi^\ell(u) = u^m$ with $m \geq 2$.
- Each letter is contained in $u$ at most once.
- Put $B = \text{alph}(u)$ and let $\pi$ be the permutation determined by the order of letters in $u$, then $\varphi$ is $(B, \pi)$-cyclic.
Thank you for your attention!