An Algorithm Enumerating All Infinite Repetitions in a D0L System

Karel Klouda[1]
karel.klouda@fit.cvut.cz

joint work with Štěpán Starosta[1]

[1]Czech Technical University in Prague

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D0L system

Definition

A D0L-system is a triplet $G = (\mathcal{A}, \varphi, w)$ where $\mathcal{A}$ is an alphabet, φ a morphism on $\mathcal{A}$, and $w \in \mathcal{A}^+$ is the axiom.
The sequence of G:

$$L(G) = \{w_0 = w, w_1 = \varphi(w_0), w_2 = \varphi(w_1), \ldots\}.$$

All factors of $w_1, w_2, \ldots$ form the language of G, denoted as $S(L(G))$.

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$G = (\{0,1,2,3,4\}, \varphi, 013)$ with $\varphi = (0310, 212, 121, 4, 3)$:

$$w_0 = 013$$
$$w_1 = 0310\,212\,4$$
$$w_2 = 031041210310\,121212121\,3$$
$$w_3 = 03104121031032121212 \cdots 0310\,2121212121 \cdots 212\,4$$
$$\vdots$$

Repetitive D0L system

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A D0L system G is repetitive if for all $k \in \mathbb{N}$ there exists a word v such that v^k is in the language of G.
It is strongly repetitive if there is a word v such that v^k is in the language of G for all k.

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Example

The D0L system $(\{0,1,2,3,4\}, \varphi, 0)$ with $\varphi = (0310, 212, 121, 4, 3)$ is strongly repetitive with $v = 21$.

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Theorem (Ehrenfeucht, Rozenberg (1983))

Every repetitive D0L system is strongly repetitive.

Our motivation

We have a tool (KK 2012) for generating all bispecial factors in a given D0L system, but it works only for non-repetitive D0L systems.

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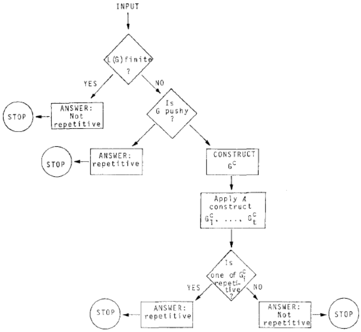
We also believe this tool can be extended to repetitive D0L systems but:

We (probably) need to know all the infinite repetitions.

Known results

Theorem (Ehrenfeucht, Rozenberg (1983))

It is decidable whether a D0L system is repetitive.



Known results

Theorem (Ehrenfeucht, Rozenberg (1983))

It is decidable whether a D0L system is repetitive.

- If the D0L system is not finite nor pushy, their procedure produces unknown number of special D0L systems.
- The original D0L system is repetitive iff one of these special D0L systems is repetitive.
- A special D0L $(\mathcal{A}, \varphi, w)$ system is repetitive iff the morphism is $(\mathcal{B}, \pi)$-cyclic for some $\mathcal{B} \subset \mathcal{A}$ and π a cyclic permutation of $\mathcal{B}$:
  - for all $b \in \mathcal{B}$, $\varphi = x\pi(x)\pi^2(x)\cdots\pi^k(x)$ with $k = |\varphi(b)| - 1$,
  - for all $b \in \mathcal{B}$: if $c = \pi(b)$ and d is the last letter of $\varphi(b)$, then $\pi(d)$ is the first letter of $\varphi(c)$.

Known results

- Mignosi, Séébold (1993): they addressed a different problem, decidability of repetitiveness is just a consequence.

- Kobayashi, Otto (2000): polynomial time algorithm, that still can be simplified.

Related problem: periodicity

Problem (ultimate periodicity)

Given a D0L system $(\mathcal{A}, \varphi, w)$ *such that* $\varphi(w) = wy$ *for some* $y \in \mathcal{A}^*$. *Is* $\varphi^\omega(w)$ *ultimately periodic?*

- First independent proofs: Harju, Linna (1986), Pansiot (1986).
- Other proofs: Honkala (2008), Halava, Harju, Kärki (WORDS 2011).

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Problem (eventual ultimate periodicity)

Given a D0L system $(\mathcal{A}, \varphi, w)$*. Is there* $i \geq 0$ *and* $p > 0$ *such that* $\varphi^p(w_i) = w_i y$ *for some* $y \in \mathcal{A}^*$ *and* $(\varphi^p)^\omega(w_i)$ *ultimately periodic?*

- First proofs: Head, Lando (1986).
- Refined proof and algorithm: Lando (1989):
  - Among other things a very simple algorithm deciding whether a fixed point of a morphism is purely periodic is presented.

Infinite periodic factor

Since the repetitions

$$(123)^{\omega}, \quad (123123)^{\omega}, \quad (312)^{\omega}, \quad (231)^{\omega}, \ldots$$

are in fact the same, we define:

Definition

Given a D0L system G, we say that v^{ω} is an infinite periodic factor of G if v is a non-empty word and $v^k \in S(L(G))$ for all integers k.

Let v be non-empty and primitive (not a power of a shorter word). We say that infinite periodic factors v^{ω} and u^{ω} are equivalent if u is a power of a conjugate of v. We denote the equivalence class containing v^{ω} by $[v]^{\omega}$.

Simplification of a morphism

Example

The morphism

$$\varphi: \quad a \to aca, \quad b \to badc, \quad c \to acab, \quad d \to adc$$

is not injective, as $\varphi(ab) = aca\,badc = acab\,adc = \varphi(cd)$. This means that $\{aca, badc, acab, adc\}$ is not a code.

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By the defect theorem, there must be three (or less) words X, Y, Z from $\{a, b, c, d\}^+$ such that

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$$g : \quad X \to b, \quad Y \to aca, \quad Z \to adc$$
$$h : \quad a \to Y, \quad b \to XZ, \quad c \to YX, \quad d \to Z$$

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We have: $\varphi = g \circ h$ and

injective simplification $h \circ g : \quad X \to XZ, \quad Y \to YYXY, \quad Z \to YZYX.$

Simplification of a morphism

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite alphabets and let $\varphi : \mathcal{A}^* \mapsto \mathcal{A}^*$ and $\psi : \mathcal{B}^* \mapsto \mathcal{B}^*$ be morphisms. We say ψ is a simplification of φ, if there exist morphisms $h : \mathcal{A}^* \mapsto \mathcal{B}^*$ and $g : \mathcal{B}^* \mapsto \mathcal{A}^*$ satisfying $g \circ h = \varphi$ and $h \circ g = \psi$ and $\#\mathcal{B} < \#\mathcal{A}$. If a morphism has no simplification, it is called elementary.

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- D0L system $(\mathcal{A}, \varphi, w)$ is repetitive iff $(\mathcal{B}, \psi, h(w))$ is repetitive.

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- Elementary morphism is injective.
- D0L system $(\mathcal{A}, \varphi, w)$ is repetitive iff $(\mathcal{B}, \psi, h(w))$ is repetitive.
- Lando (1989) + our result: There is one-to-one correspondence between infinite periodic factors of these two D0L systems.
- The construction of a simplification of a non-injective morphism can be done in polynomial time.

Graph of infinite factors

Definition

Let $G = (\mathcal{A}, \varphi, w)$ be a D0L-system. The graph of infinite periodic factors of G, denoted P_G, is a directed graph with loops allowed and defined as follows:

1. the set of vertices of P_G is the set

$$V(\mathrm{P}_G) = \{[v]^\omega \mid v^\omega \text{ is an infinite periodic factor of } S(L(G))\};$$

2. there is a directed edge from $[v]^\omega$ to $[z]^\omega$ if $\varphi(v^\omega) \in [z]^\omega$.

Obviously, the outdegree of any vertex of P_G is equal to one.

Graph of infinite factors

Lemma

If $G = (\mathcal{A}, \varphi, w)$ *is an injective D0L system, then any vertex* $[v]^\omega \in \mathrm{P}_G$ *has indegree at least* 1.

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If $G = (\mathcal{A}, \varphi, w)$ is an injective D0L system, then any vertex $[v]^\omega \in \mathrm{P}_G$ has indegree at least 1.

Corollary

If $G = (\mathcal{A}, \varphi, w)$ is an injective D0L system, then its graph of infinite periodic factors P_G is 1-regular. It other words, P_G consists of disjoint cycles.

Pushy D0L system

Definition

Given a morphism φ on $\mathcal{A}$. A letter $a \in \mathcal{A}$ is bounded if the language of $(\mathcal{A}, \varphi, a)$ is finite; $\mathcal{A}_0$ is the set of all bounded letters.

Definition

A D0L system G is pushy, if its language contains infinite number of factors over $\mathcal{A}_0$.

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Example

Consider again the D0L system $(\{0,1,2,3,4\}, \varphi, 0)$ with $\varphi = (03103, 212, 121, 4, 3)$. The bounded letters are $\mathcal{A}_0 = \{3,4\}$. The system is pushy as $(34)^k$ is a factor for all $k \in \mathbb{N}$.

Pushy D0L system: what is known

- It is decidable whether a D0L system is pushy (Ehrenfeucht, Rozenberg (1983)).
  - Pushy iff edge condition: there exist $a \in \mathcal{A}$, $k \in \mathbb{N}^+$, $v \in \mathcal{A}^*$ and $u \in \mathcal{A}_0^+$ such that $\varphi^k(a) = vau$ or $\varphi^k(a) = uav$.

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- An algorithm based on a simple graphs.
  - Graphs on unbounded letters: there is a directed edge from a to b with label u if $\varphi(a) = vbu$ (resp. $\varphi(a) = ubv$) with $v \in \mathcal{A}^*$ and $u \in \mathcal{A}_0^*$.
  - Pushy iff there is a cycle with a non empty label.

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  - Pushy iff there is a cycle with a non empty label.

Theorem (Cassaigne, Nicolas (2010))

If G is a non-erasing pushy D0L system, then there exist $K \in \mathbb{N}$ and a finite set $\mathcal{U}$ of words from $\mathcal{A}_0^+$ such that every factor from $S(L(G)) \cap \mathcal{A}_0^+$ is of one of the following three forms:

(i) w_1,

(ii) $w_1 u_1^{k_1} w_2$,

(iii) $w_1 u_1^{k_1} w_2 u_2^{k_2} w_3$,

where $u_1, u_2 \in \mathcal{U}$, $|w_j| < K$ for all $j \in \{1, 2, 3\}$, and $k_1, k_2 \in \mathbb{N}^+$.

Infinite periodic factors containing an unbounded letter

Theorem

If $[v]^\omega$ *is an infinite periodic factor of a D0L system* $G = (\mathcal{A}, \varphi, w)$ *such that* $v \notin \mathcal{A}_0^+$, *then there exist*

- u *such that* u^ω *is equivalent to* v^ω,
- $a \in \mathcal{A}$ *and* $\ell \leq \#\mathcal{A}$ *such that* u^ω *is the fixed point of* φ^ℓ *starting with* a.

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In other words: all infinite periodic factors containing an unbounded letter are purely periodic periodic points of φ*.*

The algorithm by Lando

Problem: for a morphism φ over $\mathcal{A}$, letter $a \in \mathcal{A}$ and integer ℓ such that $\varphi^\ell(a) = av$ with $v \in \mathcal{A}^+$ decide whether $(\varphi^\ell)^\infty(a)$ is purely periodic:

1. If $v \in \mathcal{A}_0^+$, return the result: $(\varphi^\ell)^\omega(a)$ is not purely periodic (but eventually periodic).
2. Apply φ^ℓ to a until $(\varphi^\ell)^k(a)$ contains two occurrences of one unbounded letter ($k < \#\mathcal{A}$).
3. If this letter is not a, then $(\varphi^\ell)^\omega(a)$ is not periodic, if it is, denote u the longest prefix containing a only as the first letter.
4. Now, $(\varphi^\ell)^\omega(a)$ is periodic if and only if $\varphi^\ell(u) = u^m$ for some integer $m \geq 2$.

A note about the algorithm by Ehrenfeucht and Rozenberg

Corollary

Let $G = (\mathcal{A}, \varphi, w)$ with φ injective and $\mathcal{A}_0 = \emptyset$. It holds that G is repetitive iff φ is $(\mathcal{B}, \pi)$-cyclic for some $\mathcal{B} \subset alph(S(L(G)))$ and π a cyclic permutation of $\mathcal{B}$.

Proof.

- There must be a primitive u and $\ell \geq 1$ such that $\varphi^\ell(u) = u^m$ with $m \geq 2$.
- Each letter is contained in u at most once.
- Put $\mathcal{B} = \text{alph}(u)$ and let π be the permutation determined by the order of letters in u, then φ is $(\mathcal{B}, \pi)$-cyclic.

□

Thank you for your attention!